Lecture 8

Lossy Media, Lorentz Force Law, Drude-Lorentz-Sommerfeld Model

8.1 Plane Waves in Lossy Conductive Media

Previously, we have derived the plane wave solution for a lossless homogeneous medium. Since the algebra of complex numbers is similar to that of real numbers,the derivation can be generalized to a conductive medium by invoking mathematical homomorphism. In other words, in a conductive, one only needs to replace the permittivity with a complex permittivity, as repeated here. When conductive loss is present, $\sigma \neq 0$, and $\mathbf{J} = \sigma \mathbf{E}$. Then generalized Ampere's law becomes

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\mathbf{E} \tag{8.1.1}$$

A complex permittivity can be defined as $\underset{\sim}{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$. Eq. (8.1.1) can be rewritten as

$$\nabla \times \mathbf{H} = j\omega\underset{\sim}{\varepsilon}\mathbf{E} \tag{8.1.2}$$

This equation is of the same form as source-free Ampere's law in the frequency domain for a lossless medium where ε is completely real. Using the same method as before, a wave solution

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{8.1.3}$$

will have the dispersion relation which is now given by

$$k_y^2 + k_y^2 + k_z^2 = \omega^2\mu\underset{\sim}{\varepsilon} \tag{8.1.4}$$

Since $\underset{\sim}{\varepsilon}$ is complex now, k_x, k_y, and k_z cannot be all real. Equation (8.1.4) has been derived previously by assuming that $\mathbf{k}$ is a real vector. When $\mathbf{k} = \mathbf{k}' - j\mathbf{k}''$ is a complex vector, some of the previous derivations for real $\mathbf{k}$ vector may not be correct here for complex $\mathbf{k}$ vector. It is also difficult to visualize a complex $\mathbf{k}$ vector that is suppose to indicate the direciton with which the wave is propagating. Here, the wave can decay and oscillate in different directions.

So again, we look at the simplified case where

$$\mathbf{E} = \hat{x} E_x(z) \tag{8.1.5}$$

so that $\nabla \cdot \mathbf{E} = \partial_x E_x(z) = 0$, and let $\mathbf{k} = \hat{z} k = \hat{z}\omega\sqrt{\mu \underset{\sim}{\varepsilon}}$. This wave is constant in the xy plane, and hence, is a plane wave. Furthermore, in this manner, we are requiring that the wave decays and propagates (or oscillates) only in the z direction. For such a simple plane wave,

$$\mathbf{E} = \hat{x}\mathbf{E}_x(z) = \hat{x} E_0 e^{-jkz} \tag{8.1.6}$$

where $k = \omega\sqrt{\mu \underset{\sim}{\varepsilon}}$, since $\mathbf{k} \cdot \mathbf{k} = k^2 = \omega^2 \mu \underset{\sim}{\varepsilon}$ is still true.

Faraday's law gives rise to

$$\mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}}{\omega\mu} = \hat{y}\frac{k E_x(z)}{\omega\mu} = \hat{y}\sqrt{\frac{\underset{\sim}{\varepsilon}}{\mu}} E_x(z) \tag{8.1.7}$$

where $\mathbf{k}$ vector is defined shortly after (8.1.5) above, and $k = \omega\sqrt{\mu \underset{\sim}{\varepsilon}}$, a complex number. It is seen that $\mathbf{H} = \hat{y} H_y$, and that

$$E_x / H_y = \sqrt{\frac{\mu}{\underset{\sim}{\varepsilon}}} \tag{8.1.8}$$

When the medium is highly conductive, $\sigma \to \infty$, and $\underset{\sim}{\varepsilon} = \epsilon - j\frac{\sigma}{\omega} \approx -j\frac{\sigma}{\omega}$. In other words, conduction current dominates over displacement current. Thus, the following approximation can be made, namely,

$$k = \omega\sqrt{\mu \underset{\sim}{\varepsilon}} \simeq \omega\sqrt{-\mu\frac{j\sigma}{\omega}} = \sqrt{-j\omega\mu\sigma} \tag{8.1.9}$$

Taking $\sqrt{-j} = \frac{1}{\sqrt{2}}(1-j)$, we have for a highly conductive medium that

$$k \simeq (1-j)\sqrt{\frac{\omega\mu\sigma}{2}} = k' - jk'' \tag{8.1.10}$$

For a plane wave, e^{-jkz}, it then becomes

$$e^{-jkz} = e^{-jk'z - k''z} \tag{8.1.11}$$

This plane wave decays exponentially in the z direction. The penetration depth of this wave is then

$$\delta = \frac{1}{k''} = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{8.1.12}$$

This distance δ, the penetration depth, is called the skin depth of a plane wave propagating in a highly lossy conductive medium where conduction current dominates over displacement current, or that $\sigma \gg \omega\varepsilon$. This happens for radio wave propagating in the saline solution of the ocean, the Earth, or wave propagating in highly conductive metal, like your induction cooker.

When the conductivity is low, namely, when the displacement current is larger than the conduction current, then $\frac{\sigma}{\omega\varepsilon} \ll 1$, we have

$$\begin{aligned} k &= \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)} = \omega\sqrt{\mu\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)} \\ &\approx \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\varepsilon}\right) = k' - jk'' \end{aligned} \tag{8.1.13}$$

The above is the approximation to k for a low conductivity medium where conduction current is much smaller than displacement current. The term $\frac{\sigma}{\omega\varepsilon}$ is called the loss tangent of a lossy medium.

In general, in a lossy medium $\varepsilon = \varepsilon' - j\varepsilon''$, and $\varepsilon''/\varepsilon'$ is called the loss tangent of the medium. It is to be noted that in the optics and physics community, $e^{-i\omega t}$ time convention is preferred. In that case, we need to do the switch $j \to -i$, and a loss medium is denoted by $\varepsilon = \varepsilon' + i\varepsilon''$.

8.2 Lorentz Force Law

The Lorentz force law is the generalization of the Coulomb's law for forces between two charges. Lorentz force law includes the presence of a magnetic field. The Lorentz force law is given by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{8.2.1}$$

The first term on the right-hand side is the electric force similar to the statement of Coulomb's law, while the second term is the magnetic force called the $\mathbf{v} \times \mathbf{B}$ force. This law can be also written in terms of the force density $\mathbf{f}$ which is the force on the charge density, instead of force on a single charge. By so doing, we arrive at

$$\mathbf{f} = \varrho\mathbf{E} + \varrho\mathbf{v} \times \mathbf{B} = \varrho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{8.2.2}$$

where ϱ is the charge density, and one can identified the current $\mathbf{J} = \varrho\mathbf{v}$.

Lorentz force law can also be derived from the integral form of Faraday's law, if one assumes that the law is applied to a moving loop intercepting a magnetic flux [61]. In other words, Lorentz force law and Faraday's law are commensurate with each other.

8.3 Drude-Lorentz-Sommerfeld Model

In the previous lecture, we have seen how loss can be introduced by having a conduction current flowing in a medium. Now that we have learnt the versatility of the frequency domain method and phasor technique, other loss mechanism can be easily introduced.

First, let us look at the simple constitutive relation where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{8.3.1}$$

We have a simple model where

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E} \tag{8.3.2}$$

where χ is the electric susceptibility. To see how $\chi(\omega)$ can be derived, we will study the Drude-Lorentz-Sommerfeld model. This is usually just known as the Drude model or the Lorentz model in many textbooks although Sommerfeld also contributed to it. These models, the Drude and Lorentz models, can be unified in one equation as shall be shown.

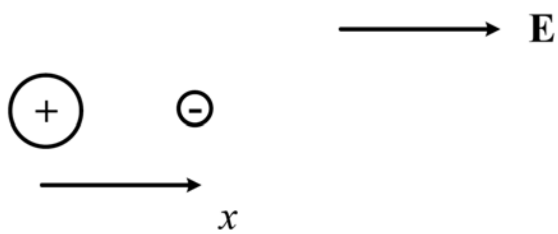

Figure 8.1: Polarization of an atom in the presence of an electric field. Here, it is assumed that the electron is weakly bound or unbound to the nucleus of the atom.

We can first start with a simple electron driven by an electric field $\mathbf{E}$ in the absence of a magnetic field $\mathbf{B}$. If the electron is free to move, then the force acting on it, from the Lorentz force law, is $-e\mathbf{E}$ where e is the charge of the electron (see Figure 8.1). Then from Newton's law, assuming a one dimensional case, it follows that

$$m_e \frac{d^2 x}{dt^2} = -eE \tag{8.3.3}$$

where the left-hand side is due to the inertial force of the mass of the electron, and the right-hand side is the electric force acting on a charge of $-e$ coulomb. Here, we assume that $\mathbf{E}$ points in the x-direction, and we neglect the vector nature of the electric field. Writing the above in the frequency domain for time-harmonic fields, and using phasor technique, one gets

$$-\omega^2 m_e x = -eE \tag{8.3.4}$$

From this, we have

$$x = \frac{e}{\omega^2 m_e} E \tag{8.3.5}$$

This, for instance, can happen in a plasma medium where the atoms are ionized, and the electrons are free to roam [62]. Hence, we assume that the positive ions are more massive, sluggish, and move very little compared to the electrons when an electric field is applied. The above equation also implies that the displacement x and the electric field E are in the same direction. It is due to the negative Coulomb force eE and the negative inertial force $\omega^2 m_e x$. A sinusoidal inertial force is actually negative to the direction of the displacement x. Moreover, this displacement x is very large when the frequency ω is low.

The dipole moment formed by the displaced electron away from the ion due to the electric field is then

$$p = -ex = -\frac{e^2}{\omega^2 m_e} E \tag{8.3.6}$$

for one electron. When there are N electrons per unit volume, the dipole moment density is then given by

$$P = Np = -\frac{Ne^2}{\omega^2 m_e} E \tag{8.3.7}$$

In general, $\mathbf{P}$ and $\mathbf{E}$ point in the same direction, and we can write

$$\mathbf{P} = -\frac{Ne^2}{\omega^2 m_e}\mathbf{E} = -\frac{{\omega_p}^2}{\omega^2}\varepsilon_0\mathbf{E} \tag{8.3.8}$$

where we have defined ${\omega_p}^2 = Ne^2/(m_e\varepsilon_0)$. Then,

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\left(1 - \frac{{\omega_p}^2}{\omega^2}\right)\mathbf{E} \tag{8.3.9}$$

In this manner, we see that the effective permittivity is

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{{\omega_p}^2}{\omega^2}\right) \tag{8.3.10}$$

What the above math is saying is that the electric field $\mathbf{E}$ induces a dipole moment density $\mathbf{P}$ that is negative to each other. This negative dipole density cancels the contribution to the electric flux from the vacuum $\epsilon_0\mathbf{E}$. For low frequency, the effective permittivity is negative, disallowing the propagation of a wave as we shall see.

Hence, $\varepsilon < 0$ if

$$\omega < \omega_p = \sqrt{N/(m_e\varepsilon_0)}e$$

Here, ω_p is the plasma frequency. Since $k = \omega\sqrt{\mu\varepsilon}$, if ε is negative, $k = -j\alpha$ becomes pure imaginary, and a wave such as e^{-jkz} decays exponentially as $e^{-\alpha z}$. This is also known as an evanescent wave. In other words, the wave cannot propagate through such a medium: Our ionosphere is such a medium. The plasma shields out electromagnetic waves that are below the plasma frequency ω_p.

Therefore, it was extremely fortuitous that Marconi, in 1901, was able to send a radio signal from Cornwall, England, to Newfoundland, Canada. Nay sayers thought his experiment would never succeed as the radio signal would propagate to outer space and never return. It is the presence of the ionosphere that bounces the radio wave back to Earth, making his experiment a resounding success and a very historic one! The experiment also heralds in the age of wireless communications.

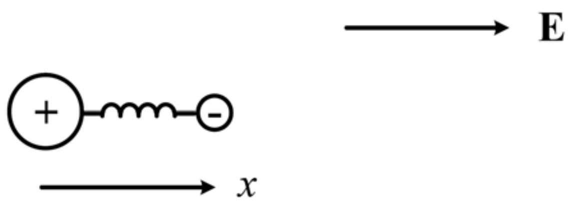

Figure 8.2: The electron is bound to the ion by an attractive force. This can be approximately modeled by a spring providing a restoring force to the electron.

The above model can be generalized to the case where the electron is bound to the ion, but the ion now provides a restoring force similar to that of a spring (see Figure 8.2), namely,

$$m_e \frac{d^2x}{dt^2} + \kappa x = -eE \tag{8.3.11}$$

We assume that the ion provides a restoring force just like Hooke's law. Again, for a time-harmonic field, (8.3.11) can be solved easily in the frequency domain to yield

$$x = \frac{e}{(\omega^2 m_e - \kappa)} E = \frac{e}{(\omega^2 - {\omega_0}^2) m_e} E \tag{8.3.12}$$

where we have defined ${\omega_0}^2 m_e = \kappa$. The above is the typical solution of a lossless harmonic oscillator (pendulum) driven by an external force, in this case the electric field. The dipole moment due to an electric field then is

$$p = -ex = -\frac{e^2}{(\omega^2 - {\omega_0}^2) m_e} E \tag{8.3.13}$$

Therefore, when the frequency is low or $\omega = 0$, this dipole moment is of the same polarity as the applied electric field E, contributing to a positive dipole moment. It contributes positively to the displacement flux $\mathbf{D}$ via $\mathbf{P}$.

Equation (8.3.11) can be generalized to the case when frictional or damping forces are involved, or that

$$m_e \frac{d^2x}{dt^2} + m_e \Gamma \frac{dx}{dt} + \kappa x = -eE \tag{8.3.14}$$

The second term on the left-hand side is a force that is proportional to the velocity dx/dt of the electron. This is the hall-mark of a frictional force. Frictional force is due to the collision of the electrons with the background ions or lattice. It is proportional to the destruction (or change) of momentum ($m_e \frac{dx}{dt}$) of an electron. In the average sense, the destruction of the momentum is given by the product of the collision frequency and the momentum. In the above, Γ has the unit of frequency, and for plasma, and conductor, it can be regarded as a collision frequency.

Solving the above in the frequency domain, one gets

$$x = \frac{e}{(\omega^2 - j\omega\Gamma - {\omega_0}^2)m_e} E \tag{8.3.15}$$

Following the same procedure in arriving at (8.3.7), we get

$$P = \frac{-Ne^2}{(\omega^2 - j\omega\Gamma - {\omega_0}^2)m_e} E \tag{8.3.16}$$

In this, one can identify that

$$\begin{aligned}\chi(\omega) &= \frac{-Ne^2}{(\omega^2 - j\omega\Gamma - {\omega_0}^2)m_e\varepsilon_0} \\ &= -\frac{{\omega_p}^2}{\omega^2 - j\omega\Gamma - {\omega_0}^2}\end{aligned} \tag{8.3.17}$$

where ω_p is as defined before. A function with the above frequency dependence is also called a Lorentzian function. It is the hallmark of a damped harmonic oscillator.

If $\Gamma = 0$, then when $\omega = \omega_0$, one sees an infinite resonance peak exhibited by the DLS model. But in the real world, $\Gamma \neq 0$, and when Γ is small, but $\omega \approx \omega_0$, then the peak value of χ is

$$\chi \approx +\frac{{\omega_p}^2}{j\omega\Gamma} = -j\frac{{\omega_p}^2}{\omega\Gamma} \tag{8.3.18}$$

χ exhibits a large negative imaginary part, the hallmark of a dissipative medium, as in the conducting medium we have previously studied.

The DLS model is a wonderful model because it can capture phenomenologically the essence of the physics of many electromagnetic media, even though it is a purely classical model.[1] It captures the resonance behavior of an atom absorbing energy from light excitation. When the light wave comes in at the correct frequency, it will excite electronic transition within an atom which can be approximately modeled as a resonator with behavior similar to that of a pendulum oscillator. This electronic resonances will be radiationally damped [33], and the damped oscillation can be modeled by $\Gamma \neq 0$.

Moreover, the above model can also be used to model molecular vibrations. In this case, the mass of the electron will be replaced by the mass of the atom involved. The damping of the molecular vibration is caused by the hindered vibration of the molecule due to interaction with other molecules [63]. The hindered rotation or vibration of water molecules when excited by microwave is the source of heat in microwave heating.

In the case of plasma, $\Gamma \neq 0$ represents the collision frequency between the free electrons and the ions, giving rise to loss. In the case of a conductor, Γ represents the collision frequency between the conduction electrons in the conduction band with the lattice of the material.[2]

[1] What we mean here is that only Newton's law has been used, and no quantum theory as yet.

[2] It is to be noted that electron has a different effective mass in a crystal lattice [64, 65], and hence, the electron mass has to be changed accordingly in the DLS model.

Also, if there is no restoring force, then $\omega_0 = 0$. This is true for sea of electron moving in the conduction band of a medium. Also, for sufficiently low frequency, the inertial force can be ignored. Thus, from (8.3.17), again we have[3]

$$\chi \approx -j \frac{{\omega_p}^2}{\omega\Gamma} \tag{8.3.19}$$

and

$$\varepsilon = \varepsilon_0(1+\chi) = \varepsilon_0 \left(1 - j\frac{{\omega_p}^2}{\omega\Gamma}\right) \tag{8.3.20}$$

We recall that for a conductive medium, we define a complex permittivity to be

$$\varepsilon = \varepsilon_0 \left(1 - j\frac{\sigma}{\omega\varepsilon_0}\right) \tag{8.3.21}$$

Comparing (8.3.20) and (8.3.21), we see that

$$\sigma = \varepsilon_0 \frac{{\omega_p}^2}{\Gamma} \tag{8.3.22}$$

The above formula for conductivity can be arrived at using collision frequency argument as is done in some textbooks [66].

As such, the DLS model is quite powerful: it can be used to explain a wide range of phenomena from very low frequency to optical frequency. The fact that $\varepsilon < 0$ can be used to explain many phenomena. The ionosphere is essentially a plasma medium described by

$$\varepsilon = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right) \tag{8.3.23}$$

Radio wave or microwave can only penetrate through this ionosphere when $\omega > \omega_p$, so that $\varepsilon > 0$. The electrons in many conductive materials can be modeled as a sea of free electrons moving about quite freely with an effective mass. As such, they behave like a plasma medium as shall be seen.

8.3.1 Frequency Dispersive Media

The DLS model shows that, except for vacuum, all media are frequency dispersive. It is prudent to digress to discuss more on the physical meaning of a frequency dispersive medium. The relationship between electric flux and electric field, in the frequency domain, still follows the formula

$$\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega) \tag{8.3.24}$$

[3]This equation is similar to (8.3.18). In both cases, collision force dominates in the equation of motion (8.3.14).

When the effective permittivity, $\varepsilon(\omega)$, is a function of frequency, it implies that the above relationship in the time domain is via convolution, viz.,

$$\mathbf{D}(t) = \varepsilon(t) \circledast \mathbf{E}(t) \tag{8.3.25}$$

Since the above represents a linear time-invariant system, it implies that an input is not followed by an instantaneous output. In other words, there is a delay between the input and the output. The reason is because an electron has a mass, and it cannot respond immediately to an applied force: or it has inertial. In other words, the system has memory of what it was before when you try to move it.

Even though the effective permittivity ϵ is a function of frequency, the frequency domain analysis we have done for a plane wave propagating in a dispersive medium still applies. Here, it also implies that different frequency components will propagate with different phase velocities through such a medium. Hence, a narrow pulse will spread in its width because different frequency components are not in phase after a short distance of travel.

Also, the Lorentzian function is great for data fitting, as many experimentally observed resonances have finite Q and a line width. The Lorentzian function models that well. If multiple resonances occur in a medium or an atom, then multi-species DLS model can be used. It is now clear that all media have to be frequency dispersive because of the finite mass of the electron and the inertial it has. In other words, there is no instantaneous response in a dielectric medium due to the finiteness of the electron mass.

Even at optical frequency, many metals, which has a sea of freely moving electrons in the conduction band, can be modeled approximately as a plasma. A metal consists of a sea of electrons in the conduction band which are not tightly bound to the ions or the lattice. Also, in optics, the inertial force due to the finiteness of the electron mass (in this case effective mass, see Figure 8.3) can be sizeable compared to other forces. Then, $\omega_0 \ll \omega$ or that the restoring force is much smaller than the inertial force, in (8.3.17), and if Γ is small, $\chi(\omega)$ resembles that of a plasma, and ε of a metal can be negative.

Table 4.2 The effective mass m_e^* of electrons in some metals.

Metal	Ag	Au	Bi	Cu	K	Li	Na	Ni	Pt	Zn
m_e^*/m_e	0.99	1.10	0.047	1.01	1.12	1.28	1.2	28	13	0.85

From *Principles of Electronic Materials and Devices, Second Edition*, S.O. Kasap (© McGraw-Hill, 2002)
http://Materials.Usask.Ca

Figure 8.3: Effective masses of electron in different metals.

8.3.2 Plasmonic Nanoparticles

When a plasmonic nanoparticle made of gold is excited by light, its response is given by (see homework assignment)

$$\Phi_R = E_0 \frac{a^3 \cos\theta}{r^2} \frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s + 2\varepsilon_0} \tag{8.3.26}$$

In a plasma, ε_s can be negative, and thus, at certain frequency, if $\varepsilon_s = -2\varepsilon_0$, then $\Phi_R \to \infty$. Gold or silver with a sea of electrons, behaves like a plasma at optical frequencies, since the inertial force in the DLS model is quite large. Therefore, when light interacts with such a particle, it can sparkle brighter than normal. This reminds us of the saying "All that glitters is not gold!" even though this saying has a different intended meaning.

Ancient Romans apparently knew about the potent effect of using gold and silver nanoparticles to enhance the reflection of light. These nanoparticles were impregnated in the glass or lacquer ware. By impregnating these particles in different media, the color of light will sparkle at different frequencies, and hence, the color of the glass emulsion can be changed (see website [67]).

Figure 8.4: Ancient Roman goblets whose laquer coating glisten better due to the presence of gold nanoparticles. Gold or silver at optical frequencies behaves like plasma (courtesy of Smithsonian.com).